# Theory of Lévy matrices

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We investigate the statistical properties of the spectrum of large symmetrical matrices with each element $H_{ij}$ chosen according to a broad distribution $\rho(H)$ decaying for large $H$ as $H^{-1-\mu}$. For $\mu>2$, $\langle H^2\rangle$ is finite and the well known Gaussian orthogonal ensemble (GOE) results are recovered. When $\mu<2$, the semicircular law is replaced by a density which extends over the whole energy axis. Furthermore, while all states are extended in the case of GOE matrices, we show numerically and analytically that *two* mobility edges appear, separating extended from localized states, with an intermediate "mixed" phase in between. The unusual nature of these localized states is discussed.

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## I. INTRODUCTION

The theory of random matrices is of paramount importance in several fields of physics: nuclear physics (where they were first introduced), quantum chaos, localization and mesoscopic conductors, spin glasses, random surfaces, and string theory [1–6]. The main achievement of the theory of random matrices is to establish *universal* results, on, e.g., the density of states or the level spacing distribution, which are independent of the particular realization of the randomness. These results provide, in a sense, the generalization to matrices of the usual central limit theorem, since they are found to hold under rather mild assumptions on the statistical distribution of each element: only the symmetry group of the matrices (orthogonal, unitary, symplectic) is of importance. For instance, in the case of real symmetric $N\times N$ random matrices with elements distributed independently according to a law with a finite variance, the distribution of eigenvalues obeys (in the large $N$ limit) the semicircle law and the distribution of level spacing is unique and well described by the Wigner surmise: they enter the universality class of Gaussian orthogonal ensemble (GOE) matrices (see [6] for a recent discussion). It is, however, well known that the central limit theorem for sums or random variables has to be modified when the variance of these variables is infinite: new "universality classes" appear and are described by the theorems of Lévy and Gnedenko [7,8]. In this paper, we consider an analogous extension of the theorems on random matrices to the case where the distribution of the elements are of infinite variance. For definiteness, we shall deal with symmetric matrices $H_{ij}=H_{ji}$ otherwise distributed independently according to $P(H_{ij})$ with

$$P(H_{ij})\underset{H_{ij}\to\infty}{\sim}\frac{H_0^{\mu}}{|H_{ij}|^{1+\mu}}\ , \tag{1}$$

where $H_0$ is the typical order of magnitude of $H_{ij}$ and $0<\mu<2$: $P(H_{ij})$ cannot be normalized if $\mu\le 0$, and the distribution has a finite variance when $\mu>2$ in which case, as will be recalled below, the matrix enters the GOE case. We shall call this set of matrices "Lévy matrices" which is qualitatively speaking the complementary of GOE in the sense of the limit theorems quoted above. It can also be compared to sparse random matrices [9–11], which correspond, in a sense, to $\mu=0$.

The statistical properties of Lévy matrices is an interesting mathematical problem in its own right; it is, however, not devoid of physical interest for the following reasons.

(1) Contrarily the GOE ensemble, where statistical rotational invariance ensures that all eigenvectors are extended (except possibly near the edges of the spectrum), we will see below that a mobility edge appears within the spectrum, separating high-energy "localized" states from low-energy extended states. We will argue that the level spacing distribution is not universal (i.e., not Poissonian) within this localized phase, resulting from the usual nature of the localized states, which we will discuss in detail. In particular, two different localization criteria lead to different mobility edges: a "mixed phase" appears where the states are both localized and extended.

(2) Broad distribution of matrix elements can indeed occur, for instead, in a disordered electronic system. Let us consider such a system in a space of dimensionality $d$ in a tight-binding approach, with a hopping element $V$ between sites which is a function of their mutual distance and decays as a power law $V(R)\sim R^{-\gamma}$ (see, e.g., [12]). The probability to find a site at a distance $R$ from an arbitrary reference site is proportional to the surface of the sphere of radius $R$: $P(R)=R^{d-1}\rho/N$ where $N$ is the total number of sites and $\rho$ their density. Then the distribu-

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tion of "couplings" $V$ is obtained from $P(V)dV = P(R)dR$, and thus $P(V)=\rho/(NV^{1+\mu})$ with $\mu=d/\gamma$. Thus the statistics of the Hamiltonian elements (written in the basis of the sites) is indeed given by Eq. (1) with $H_0=(\rho/N)^{1/\mu}$; we shall thus often refer to this system in order to obtain some precise physical interpretations of the different quantities which we will encounter.

We shall see later than one indeed has to choose $H_0 \sim 1/N^{1/\mu}$ in order to get a sensible limit for the density of states in the limit of infinite matrices. However, the Hamiltonian of the putative electronic system is not exactly a "Lévy matrix" for finite $d$. Indeed, even if the positions of the sites are random, their mutual distances and thus the couplings, are correlated, correlations which are neglected in the Lévy matrices. In the case of randomness in the position of the sites, the total number of degrees of freedom is $Nd$ whereas it is of order $N^2$ in the case of Lévy matrices. The effective dimension of the problem is then of order $N$ and is infinite for infinite matrices. However, Lévy matrices should capture some features of this class of random systems, at least those governed by the largest couplings (see, for example, the tail of the density of states in Sec. II) which are rare enough to be considered uncorrelated. Although it is generally thought that there is not transition between localized and extended states on a fully connected lattice, we have already mentioned that Lévy matrices in fact present such a transition. Consequently, this system could be a good starting point to study localization of interacting electrons in infinite dimension within the framework proposed in [13,14].

Another example of a random system with power-law interactions is that of spin glass with dipolar or Ruderman-Kittel-Kasuya-Yosida (RKKY) interactions $|V(R)| \sim R^{-3}$, corresponding (in $d=3$) to $\mu=1$: the eigenvalues and eigenvectors of the Lévy matrix should correspond to the eigenfrequencies and eigenmodes of the spin waves around the ground state of the associated system [4,15] (this was already noted by Anderson in his original article [16]). Experimental papers discussing these localized high frequency magnons can be found in [17], and references therein.

This paper is organized as follows: In Sec. II, we define precisely the model studied, and we establish a recursion relation for the one-point Green function. The study of this equation allows us to determine the tail of the distribution of $G_{ii}$ and the distribution of eigenvalues (or density of states) is obtained analytically as the solution of integral relations. We give a precise interpretation of the tail of this distribution, and the analytical results obtained are corroborated by careful numerical simulations (Sec. II). We then analyze the recursion relation for the derivative of $G_{ii}$ with respect to the energy and obtain analytically a transition between localized and extended states (Sec. III). We also present in this third part several numerical approaches to study this transition (inverse participation ratio, distribution of the level spacings), which point towards the existence of a second transition, distinct from the previous one, and allow us to specify the structure of the localized states. From a technical point of view, our approach is somewhat new since we choose to keep the energy *real* (as opposed to the usual procedure of adding a small imaginary part). The density of states, for example, is obtained through the analysis of the distribution of a real quantity, rather than the (well defined) limit of a complex number. This point of view is, we believe, rather transparent and physical.

## II. THE MODEL AND THE RECURSION RELATION

### A. The model

The set of the Lévy matrices are the real symmetric matrices $H$ of size $N$ with elements $H_{ij}$ distributed independently according to a symmetric power law, given by Eq. (1) above. The distribution $P$ has, for $\mu<2$, an infinite variance. The most important qualitative feature of such distributions is the statistical "hierarchy" of the matrix elements: a typical element is of order $H_0$, the largest element of a row containing $N$ terms is of order $H_0 N^{1/\mu}$ [8], and the largest element of the whole matrix ($N^2$ terms) is of order $H_0 N^{2/\mu}$. This hierarchy becomes more and more pronounced as $\mu$ goes to zero. In this sense, $\mu=0$ corresponds to sparse matrices.

In order for the distribution of eigenvalues to reach a stable form in the limit of infinite size matrices, the largest element of a row must typically be of order 1, thus imposing that $H_0 \propto N^{-1/\mu}$. We will see *a posteriori* that it is the correct scaling; note in particular that for $\mu=2$ one recovers the usual GOE scaling. In the following, we shall set $H_0 \equiv N^{-1/\mu}$.

### B. The recursion relation

In order to obtain some results on the distribution of eigenvalues or to discuss the existence of a transition, we studied the one-point Green function defined as

$$G_{ii}(z)=(z-H)^{-1}_{ii}=\frac{1}{N}\sum_{\alpha=1}^{N}\frac{\langle i|\alpha\rangle^2}{z-E_\alpha}\ , \tag{2}$$

where $|i\rangle$ is the canonical (site) basis of $H$, and $|\alpha\rangle$ and $E_\alpha$ the eigenvectors and eigenvalues of $H$. We can write a recursion relation for this quantity in the following way. Let us consider a symmetric real matrix $H$ of size $N$, and its associated "resolvent" $G=(z-H)^{-1}$. One can express $G_{ii}$ as a Gaussian integral over auxiliary fields $\phi_i$,

$$G_{ij}^N=\frac{\int\prod_{k=1}^{N}d\phi_k\phi_i\phi_j\exp\left[-\frac{1}{2}\sum_{k,l=1}^{N}\phi_k(z-H)_{kj}\phi_l\right]}{\int\prod_{k=1}^{N}d\phi_k\exp\left[-\frac{1}{2}\sum_{k,l=1}^{N}\phi_k(z-H)_{kj}\phi_l\right]}\ . \tag{3}$$

In the spirit of the "cavity" method [18], one can add an $(N+1)$th row and its symmetric column to $H^N$, which we call $\{H_{0i}\}$. In terms of a tight-binding model, this corresponds to adding an $(N+1)$th site 0 to the system. The size of the matrix is $N+1$ and Eq. (3) for $G_{00}^{N+1}$ now reads

$$G_{00}^{N+1}=\frac{\int\prod_{k=0}^{N}d\phi_k\phi_0^2\exp\left[-\frac{1}{2}\sum_{k,l=0}^{N}\phi_k(z-H)_{kj}\phi_l\right]}{\int\prod_{k=0}^{N}d\phi_k\exp\left[-\frac{1}{2}\sum_{k,l=0}^{N}\phi_k(z-H)_{kj}\phi_l\right]}\ . \tag{4}$$

One can integrate over auxiliary fields $\{\phi_k\}_{k=1,N}$:

$$G_{00}^{N+1}=\frac{\int d\phi_0\phi_0^2\exp\left[-\frac{1}{2}\sum_{k,l=1}^{N}H_{0i}G_{ij}^{N}H_{j0}\phi_0^2-\frac{1}{2}(z-H_{00})\phi_0^2\right]}{\int d\phi_0\exp\left[-\frac{1}{2}\sum_{k,l=1}^{N}H_{0i}G_{ij}^{N}H_{j0}\phi_0^2-\frac{1}{2}(z-H_{00})\phi_0^2\right]}, \tag{5}$$

which is a Gaussian integral. Integrating over $\phi_0$, one finds the result

$$\frac{1}{G_{00}^{N+1}(z)}=z-H_{00}-\sum_{i,j=1}^{N}H_{0i}G_{ij}^{N}(z)H_{j0}\ , \tag{6}$$

which is the central recursion relation which we shall exploit in the following. (Note that this relation can be obtained without Gaussian integrals through adequate algebraic manipulations.) This general equation is simplified in the case of Lévy matrices where the elements are typically of order $1/N^{1/\mu}$. First of all, $H_{00}$ is of order $1/N^{1/\mu}$ and can be neglected compared to $z$. One can then separate, in the right hand side of Eq. (6), the diagonal terms and the off-diagonal terms. Using the results of Appendix A, the latter is of order $[(1/N^2)\sum_{(i\neq j)=1}^{N}|G_{ij}|^{\mu}]^{1/\mu}$. One can write the equivalent of Eq. (6) for the off-diagonal terms $G_{ij}$: one self-consistently finds that they are equalitatively distributed according to the same law as $H_{ij}$. The above off-diagonal sum is then also of order $N^{-1/\mu}$ and goes to 0 as $N$ goes to infinity. To lowest order, we thus find

$$S_0(z)=\sum_{i=1}^{N}H_{0i}^2G_{ii}^{N}(z)\ , \tag{7}$$

where we have defined the self-energy $S_0(z)\equiv z-1/G_{00}^{N+1}(z)$ (from now on, we will forget the index $N$ and bear in mind that $i$ is related to $H^N$ and 0 to $H^{N+1}$). The most interesting feature of this equation is that the couplings $H_{0i}$ and the terms $G_{ii}^{N}$ are by construction uncorrelated, and hence one may calculate the self-energy using central limit theorems. One must, however, not forget that $G_{ii}^{N+1}$ is different from $G_{ii}^{N}$. Two remarks are now in order.

One should note that only $H_{0i}^2$ appears in Eq. (7), showing that a possible asymmetry in the distribution law (1) is irrelevant: all the properties obtained below would still be valid if we had considered—say—*positive* Lévy matrices (we have checked this somewhat surprising statement numerically).

(2) Equation (7) is very similar to the starting equation in the article of Abouchacra, Anderson, and Thouless [19], see also [20] for the problem of localization on a Cayley tree with a diagonal disorder. The mean difference is that the sum appearing in Eq. (7) is over $K$ terms in the case of the Cayley tree, $K$ being the connectivity of the tree, rather than $N$ terms in our case. Our equation is thus easier to solve since the limit $N\to\infty$ will allow us to use central limit theorems.

### C. Density of states

There are two ways of interpreting Eq. (7). If $\epsilon$ is the imaginary part of $z$, the standard way is to take $\epsilon$ small but positive. Then the trace of $G$, $T(z)\equiv(1/N)\sum_{i=1}^{N}G_{ii}=(1/N)\sum_{\alpha=1}^{N}[1/(z-E_\alpha)]$ is complex but tends to a *finite* limit as $N$ goes to infinity. The density of states $\rho(z)$ is then given by

$$\rho(z)=\frac{1}{\pi}\lim_{\epsilon\to0^+}\lim_{N\to\infty}\mathrm{Im}T(z-i\epsilon)\ . \tag{8}$$

The second way of looking at Eq. (7) is to consider that $\epsilon=0$. $T$ then fluctuates even in the limit of infinite size matrices, because the denominator $z-E_\alpha$ can become arbitrarily small (although nonzero since the number of eigenvalues is denumerable). One can argue that the tail of the distribution of $T$ decays as $\rho(z)/T^2$ $(T\to\infty)$, since the probability that $z-E_\alpha$ is very small (corresponding to large $T$) is finite and equals $\rho(z)$. This argument will be made more precise below.

In the case of matrices with elements distributed with a finite variance $\overline{H_{0i}^2}=\sigma/N$ $(\mu>2)$, Eq. (7) can easily be solved using the first method: if $\epsilon$ is positive, $G_{ii}$ is bounded by $1/\epsilon$, and the self-energy $S(z)$ is calculated using the central limit theorem: it tends to its mean value $\overline{H_{0i}^2}\sum_{i=1}^{N}G_{ii}=\sigma T$. Since for $\epsilon>0$ $T$ is self-averaging when $N\to\infty$, i.e., $T=\overline{G_{ii}}$, then $G_{00}$ becomes independent of the disorder and is equal to $T$, thus leading to a second order equation in $T$:

$$\sigma T^2-zT+1=0\ , \tag{9}$$

which has a complex solution when $|z|<2\sigma$ and thus, from Eq. (8), the density of states obeys the semicircle law: $\rho(z)=1/(2\pi)\sqrt{4-z^2}/\sigma$. The only hypothesis is that the variance of $H_{0i}^2$ is finite.

The second method is more adapted to the case of Lévy matrices. (Note that when applied to the GOE case, the following method fails because the sum $\sum_{i=1}^{N}G_{ii}H_{0i}^2$, with $G_{ii}$ seen as a "$\mu=1$" random variable does not obey the generalized central limit theorem since strong correlations exist between the $G_{ii}$). Since the self-energy is not self-averaging, the variance $\overline{H_{0i}^2}$ being infinite generalized central limit theorem tells us that the distribution of the self-energy tends to a stable Lévy distribution (see Appendix A): $P_S(S)=L_{\mu/2}^{C(z),\beta(z)}(S)$. It is characterized by three parameters $\mu$, $C$, and $\beta$, with

$$C(z)=\frac{1}{N}\sum_{i=1}^{N}|G_{ii}|^{\mu/2}\ , \tag{10a}$$

$$\beta(z)=\frac{1}{N}\sum_{i=1}^{N}\mathrm{sgn}(G_{ii})\ , \tag{10b}$$

$C$ is a "generalized variance," and $\beta$ is a parameter measuring the asymmetry of the distribution. Equation (7) now fixes the distribution of the real quantity $G_{00}$:

$$P_G(G_{00})=\frac{1}{G_{00}^2}P_S\left[z-\frac{1}{G_{00}}\right] . \tag{11}$$

We need a hypothesis to close this recursion relation: we will suppose that the distribution of $G_{ii}^{N+1}$ is the same as the distribution of $G_{ii}^{N}$ when $N$ goes to infinity. This natural hypothesis allows us to write self-consistent equations on $C$ and $\beta$ as

$$C(z)=\int_{-\infty}^{+\infty} dG\,|G|^{\mu/2-2}L_{\mu/2}^{C(z),\beta(z)}\left[z-\frac{1}{G}\right] , \tag{12a}$$

$$\beta(z)=\int_{z}^{+\infty} dG\,L_{\mu/2}^{C(z),\beta(z)}\left[z-\frac{1}{G}\right] . \tag{12b}$$

These integral equations have a unique solution for all $z$, which we have determined numerically. Hence Eqs. (11) and (12) completely determine the distribution of $G_{00}$. We now have to relate this distribution to the density of states. Let us consider $N$ finite. The mean spacing between eigenvalues around an energy $z$ is $\rho(z)/N$. $G_{00}$ will be very large if there exists some $\alpha$ such that $|z-E_\alpha| \ll \rho(z)/N$. Then the sum is dominated by this particular term $\langle i|\alpha\rangle^2/(z-E_\alpha)$ (a more detailed discussion in the case of localized states is given in Appendix B). The tail of the distribution of $G_{00}$ is then given by

$$\begin{aligned}P(G_{00})&=N\int\int dE\,\rho(E)dw\,P_w(w|E)\delta\left[G_{00}-\frac{w}{z-E}\right]\\&=\frac{N}{G_{00}^2}\int dw\,wP_w\left[w\left|z-\frac{w}{G_{00}}\right.\right]\rho\left[z-\frac{w}{G_{00}}\right]\\&\to_{G_{00}\to\infty}\frac{\rho(z)}{G_{00}^2} ,\end{aligned} \tag{13}$$

where we have used the normalization condition $\int_0^1 dw\,wP(w|z)=N^{-1}$. Comparing Eqs. (7) and (13), we find the relation for the density of states:

$$\rho(z)=P_S(z) . \tag{14}$$

This equation is a central result of our paper and is very general, since the equations (7) and (13) do not depend on the particular system considered. In the case of Lévy matrices, Eq. (14) becomes

$$\rho(z)=L_{\mu/2}^{C(z),\beta(z)}(z) , \tag{15}$$

where $C$ and $\beta$ are determined self-consistently using the integral equations (12). Note that $\rho(z)$ is *not* a Lévy distribution since $C$ and $\beta$ are themselves functions of $z$.

It is easy to show that $\rho(z)$ is symmetric around $z=0$ and is not bounded. Equations (12) can be solved analytically for $z$ small and for $z$ going to infinity. In this case, we find

$$\rho(z\to 0)=a_\mu-b_\mu z^2 , \tag{16a}$$

where

$$a_\mu=\left[\frac{\sin(\pi\mu/2)}{\pi}\right]^{1/\mu}[\mu/2\Gamma(\mu/2)]^{2/\mu}\frac{2}{\pi\mu}\Gamma(2/\mu)$$

and $b_\mu$ is another $\mu$ dependent number, and

$$\rho(z\to\infty)=\frac{c_\mu}{z^{1+\mu}} , \tag{16b}$$

where $c_\mu=NH_0^\mu\equiv 1$ for the normalization chosen in this paper. The density of states for all values of $z$ is obtained numerically, and is shown in Fig. 1. A simple interpretation of the tail of the distribution can be given by looking at the distribution of the largest element $H_{\max}$ of—say—the $i$th row of the matrix $H$:

$$\begin{aligned}Pr(|H_{\max}|=X)&=\frac{d}{dX}\left[1-\int_X^{+\infty}dH\frac{H_0^\mu}{NH^{1+\mu}}\right]^N\\&=\frac{NH_0^\mu}{X^{1+\mu}}\exp\left[-\frac{NH_0^\mu}{\mu X^\mu}\right] .\end{aligned} \tag{17}$$

We see that the tail of this distribution is *exactly the same* as (16b). If $H_{\max}=H_{ij}\gg 1$ we can make the assumption that all the other elements of the row are small enough to assume in a first approximation that they equal 0. Then $H|i\rangle=H_{\max}|j\rangle$ and since the matrix is symmetric, $H|j\rangle=H_{\max}|i\rangle$. Hence the vector $1/2(|i\rangle+|j\rangle)$ is a near eigenvector with eigenvalue $H_{\max}$: the large eigenvalues of the matrix $H$ correspond to very large elements of the matrix. We will see in the next section an argument which confirms this interpretation.

We have performed numerical simulations diagonalizing Lévy matrices. We determined the histogram of eigenvalues for matrices of different sizes (up to $N=1500$). By careful extrapolation to infinite size (Fig. 1), we obtained the values of $\rho(0)$ and $z^{1+\mu}\rho(z)$ for large $z$, and found very good agreement with the theoretical prediction [Eqs. (16)].

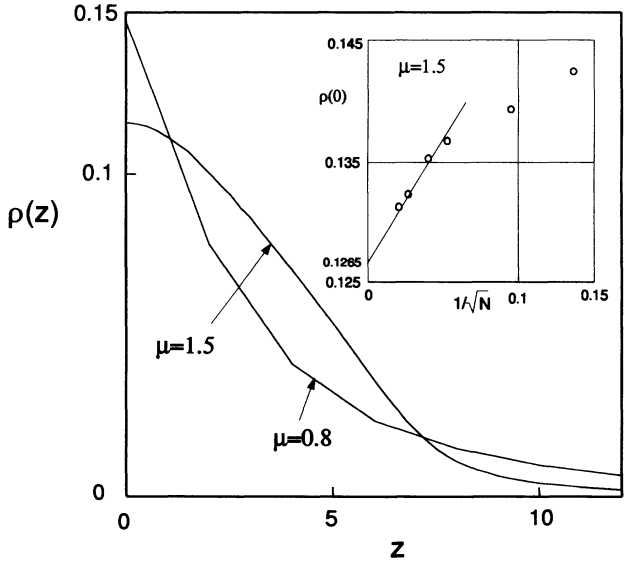


FIG. 1. Shape of the density of states $\rho(z)$ obtained by solving Eqs. (12a) and (12b) for $\mu=1.5$ and 0.8. Inset: Value of $\rho(z=0)$ for different sizes, up to $N=1500$, for $\mu=1.5$. The theoretical prediction is $\rho(0)=0.1253$.

## III. NATURE OF EIGENVECTORS

If the above interpretation of the tail of the density of states is correct, then the very high-"energy" eigenvectors are certainly localized on two vectors of the canonical basis. This suggests that a transition between localized and extended states could exist for a certain value of the energy. As we shall detail now, different localization criteria can be investigated, leading to different answers.

### A. Inverse participation ratio

A simple way to see if an eigenvector $|\alpha\rangle$ is localized on the initial basis $|i\rangle$ is to calculate its "inverse participation ratio" defined as

$$Y_\alpha = \sum_{i=1}^{N} \langle \alpha | i \rangle^4 = \sum_{i=1}^{N} w_{\alpha i}^2 , \tag{18}$$

where $w_{\alpha i} \equiv \langle \alpha | i \rangle^2$ is the weight of site $i$ in the state $\alpha$. ($\sum_{i=1}^{N} w_{\alpha i} = 1$ because the state is normalized). Two extreme cases can be analyzed. If $|\alpha\rangle$ is completely delocalized with each $w_{\alpha i}$ of order $1/N$, then from Eq. (18), $Y_\alpha$ is of order $1/N$ and goes to 0 as $N$ goes to infinity. If on the other hand $|\alpha\rangle$ is strongly localized on $p$ sites, then the sum essentially runs over $p$ terms, each one of them being of order $1/p$. $Y_\alpha$ is thus of order $1/p$ and remains not zero as $N$ goes to infinity. It is called inverse participation ratio since it is inverse of the number of sites participating to state $\alpha$, i.e., on which $\alpha$ has a nonzero projection. A natural definition of an extended state is that its $Y_\alpha$ should be zero (and conversely not zero for a localized state). It appears, however, that this quantity is not self-averaging (Appendix A), and hence a way to characterize the transition $z_c$ is to calculate the mean value of $Y_\alpha$ for a given energy $z$:

$$\bar{Y}(z) = \lim_{N\to\infty} \frac{1}{N} \rho(z) \sum_{\alpha=1}^{N} \delta(E_\alpha - z) Y_\alpha \begin{cases} = 0 & \text{for } z < z_c \\ > 0 & \text{for } z > z_c \end{cases} . \tag{19}$$

There exists a simple interpretation of $\bar{Y}(z)$ for the associated tight-binding electronic Hamiltonian. If an electron is prepared at time $t=0$ on a particular site $|m\rangle$, it will diffuse over the lattice at later times. The probability for the electron to be on site $|m\rangle$ at time $t$ is

$$P_m(t) = \sum_{\alpha\alpha'} \exp[i(E_\alpha - E_{\alpha'})t/\hbar] \langle m|\alpha\rangle^2 \langle m|\alpha'\rangle^2 \tag{20}$$

and the fraction of the total time spent on $|m\rangle$ is $f_m = \lim_{t\to\infty} (1/t) \int_0^t dt' P_m(t')$. A natural criterion of localization in terms of $f_m$ is the following: if $f_m$ is not zero the electron spends a finite fraction of the total time on the site $|m\rangle$ and is thus localized "around" $|m\rangle$. If $f_m$ is zero, the electron is delocalized over the lattice. Following [21], we can write $f_m$ as $f_m = \sum_{\alpha=1}^{N} \langle m|\alpha\rangle^4$. Note that $Y_\alpha$ is the sum over the sites of $w_{\alpha i}^2$ *for a fixed eigenstate* $|\alpha\rangle$, while $f_m$ is the sum *for a fixed site* $|m\rangle$ over the eigenstates $|\alpha\rangle$. The two quantities are of course related through the equation $\sum_i f_i \equiv \sum_\alpha Y_\alpha$. Thus one finds that $(1/N)\sum_i f_i = \int dz\, \rho(z) \bar{Y}(z)$. Hence, if $\bar{Y}(z) > 0$ for some $z > z_c$, the mean fraction of time spent on the "starting" site is finite. Another, more direct, interpretation of the localization criterion $\bar{Y}(z) > 0$ will be discussed in the next section.

We have studied $\bar{Y}(z)$ numerically by analyzing the eigenvectors of $N \times N$ Lévy matrices. We calculated $\overline{Y_N}(z)$ for energies $E_\alpha$ in a certain small interval, and for different $N$ (from 200 to 1500). Extrapolating to infinite $N$ (the corrections are proportional to $1/N$), we obtain the curve drawn in Fig. 2 (for $\mu = 1.5$). The "size" $p$ of the localized states diverges as $(z - z_c)^{-2}$ for all $\mu$—this is at variance with the results found on the Cayley tree, where $Y$ is discontinuous at the transition [22]. $\bar{Y}(z)$ furthermore tends to $\frac{1}{2}$ when $z$ goes to infinity. This last result confirms our interpretation on the tail of the density of states: we argued in Sec. II that $1/2(|i\rangle + |j\rangle)$ should be a near eigenvector if $H_{ij}$ is a very large element of the matrix $H$. The inverse participation ratio of such a vector is indeed $\frac{1}{2}$.

The transition line in the $(\mu, z)$ plane is reported qualitatively in Fig. 3. We find a finite energy threshold $z_c$ between localized and extended states for $2 > \mu > 1$, which diverges for $\mu \to 2$ (all states are extended in the GOE limit) and decreases with $\mu$ ($z_c = 4.2$ for $\mu = 1.5$, $z_c \simeq 1.5$ for $\mu = 1.1$) to reach $z_c = 0$ for $\mu < 1$. Note, however, that $\bar{Y}(z=0) = 0$ for all $\mu$: the states with zero energy are always extended in this model.

### B. Nature of localized states

Instead of looking at $Y_\alpha$ which is the mean value (over the sites $i$) of $w_{\alpha i}^2$, one can study the full distribution $P(w)$ of $w_{\alpha i}$ over the sites. Using the same numerical treatment as above, we find that the following two points hold.

(1) For $\mu > 1$ and $z < z_c$, the distributions $P_N(w)$ for different $N$ take the scaled form $P_N(w) = N\Pi(Nw)$, which means that all weights are of order $N^{-1}$. $\Pi(u)$ can be very satisfactorily fitted by the following form (Fig. 4):

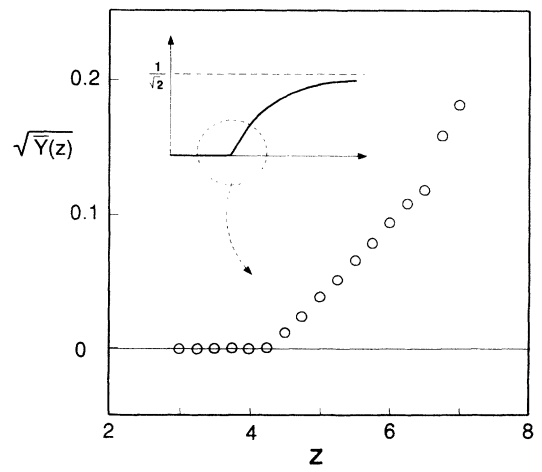


FIG. 2. Behavior of the inverse participation ratio near the transition $z_c$, for $\mu = 1.5$, showing that $\bar{Y}(z) \simeq (z - z_c)^2$. Inset: Behavior of $\sqrt{\bar{Y}(z)}$ over the full range of $z$ (qualitative). Note that $\bar{Y}(z \to \infty) = \frac{1}{2}$ for reasons explained in the text.

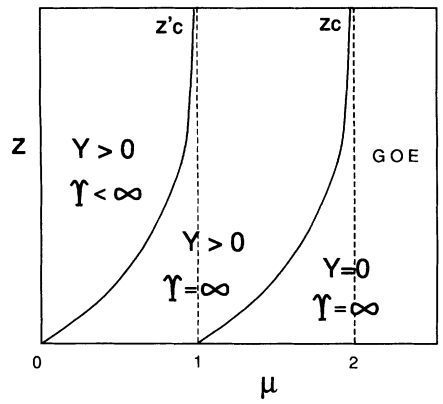


FIG. 3. Phase diagram in the plane $(z,\mu)$ exhibiting three "phases": "extended" states ($Y=0,\Upsilon=\infty$), for sufficiently large $\mu$ (or small $z$), "localized" ($Y>0,v<\infty$) for small $\mu$ (large $z$), and "mixed"—i.e., exhibiting both localized ($Y\geq 0$) and extended ($\Upsilon=\infty$) features. Note that the mixed phase is qualitatively different for $\mu<1$ and $\mu>1$ [see $P(w)$ [see Eq. (21)] and the level spacing distribution $P_s(x)$].

$$\Pi(u)=\frac{b}{u_0\Gamma[(1-a)/b]}\left[\frac{u_0}{u}\right]^a \exp-\left[\frac{u}{u_0}\right]^b , \qquad (21)$$

with, for $\mu=1.5$, $a=0.59\pm0.05$, $b=0.81\pm0.05$, and $u_0=2.0\pm0.15$. Note that the usual Porter-Thomas [23] distribution corresponds to $a_{\rm PT}=\frac{1}{2}$, $b_{\rm PT}=1$, and $u_0=1$. We have, however, not found a proper justification of Eq. (21).

This distribution changes qualitatively for $z>z_c$ (see Fig. 4) and is found to decay *as a power law*:

$$P(w)\sim\frac{c(z)}{Nw^{1+\nu}} . \qquad (22)$$

Our data suggest that $\nu\simeq\mu/2$ for all the values of $\mu$ and depends weakly on $z$. $c(z)$ is a decreasing function of $z$ and goes to zero as $z$ goes to infinity.

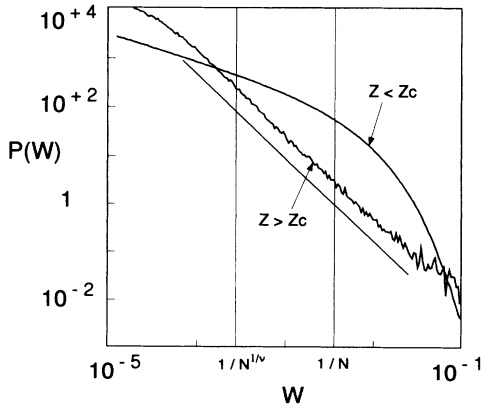


FIG. 4. Weight distribution $P(w)$ for $\mu=1.5$, $N=300$. For $z<z_c$, $P(w)$ is well fitted by Eq. (21) (fit indistinguishable from the data), while for $z>z_c$, $P(w)$ is a power law [see Eq. (22)] with $\nu=0.7\simeq\mu/2$. We have shown the scales $N^{-1/\nu}$ and $N^{-1}$ for reference.

(2) For $\mu<1$, $P(w)$ is still given by a power law [Eq. (22)] but only above a certain energy $z_c'(\mu)$ which we shall discuss below: see Fig. 5. The exponent $\nu$ is again of the order of $\mu/2$, although slightly larger: this will be confirmed by our analytical calculation below. For $z<z_c'(\mu)$, on the other hand, $P_N(w)$ has weight both in the region $w\simeq N^{-1}$ and in the region $w\simeq 1$ [giving a nonzero contribution to $\bar{Y}(z)$]. In the region $w\simeq N^{-1}$, the above scaling $P_N(w)=N\Pi(Nw)$ still holds (see Fig. 6), with the same parametrization of $\Pi(u)$. The exponents $a$ and $b$ for $\mu=0.6$ are found to be $a=0.71\pm0.05$ and $b=0.29\pm0.05$, while $u_0\simeq 0.12$.

We can use the language of the tight-binding model in order to give a physical interpretation to the power-law distribution of $w$, Eq. (22). Consider, in a space of dimension $d$, a certain localized eigenstate $|\alpha\rangle$ centered on $R=0$. $w_{\alpha i}$ is function of the distance $R$ between the site $|i\rangle$ and the origin. If the state is exponentially localized, $w(R)\sim\exp(-R)$. Moreover, the probability to find a site at a distance $R$ is proportional to the surface of the sphere of radius $R$: $P(R)dR=\rho R^{d-1}dR/N$. Hence the probability of a given weight $w$ for exponentially localized states in $d$ dimensions is $P(w)\sim 1/[Nw\ln^{d-1}(w)]$. In our case, the distribution of $w$ is a power law, which corresponds in real space to $w(R)\sim 1/R^{\gamma}$ with $\gamma=d/\nu$. Interestingly, the localized states of our Lévy matrices are the equivalent of *algebraically localized states*. In the limit of infinitely strong hierarchy of couplings $\mu=0$, one finds, using $\nu=\mu/2,\gamma=\infty$, corresponding to exponentially localized states.

Equation (22) suggests furthermore a second interpretation of $Y>0$ for localized states: one can show (see Appendix B) that if the $\{w_{\alpha i}\}$ are distributed according to (22), then $\bar{Y}(z)$ will be nonzero if $\nu<1$, or equivalently if $\gamma>d$. This precisely corresponds to the condition under which the states can be normalized, i.e., $\int d^{dR}w(R)<\infty$ for $\gamma>d$.

### C. Analytical results

In the same spirit as for the distribution of eigenvalues (Sec. II C), there are two ways to look at the transition us-

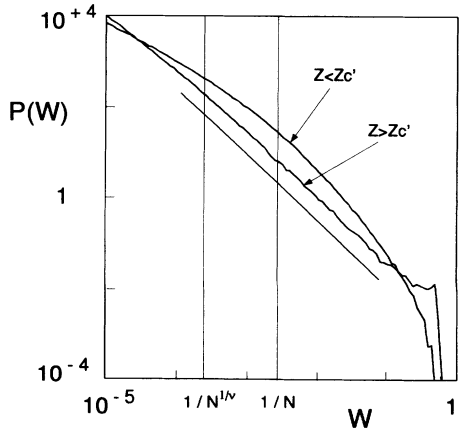


FIG. 5. Same as for Fig. 4, but for $\mu=0.6<1$, $N=400$. For $z=12>z_c'$, we find $\nu=0.42>\mu/2$.

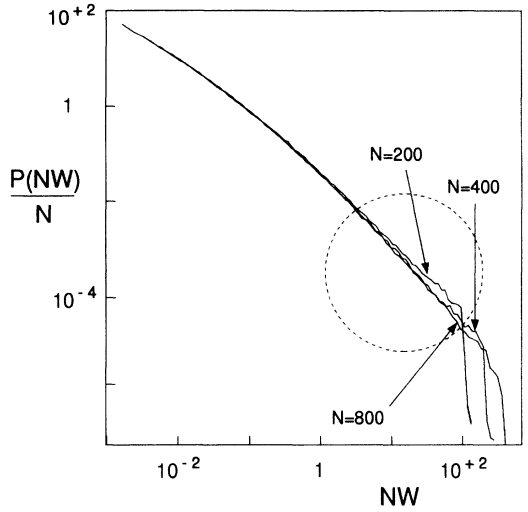


FIG. 6. Scaling of $P_n(w)$ for different sizes $N$ as $N\Pi(Nw)$, for $\mu=0.6$ and $z=4$. Note that the departure from perfect scaling for large $Nw$ (dashed circle) corresponds to the fact that a finite fraction of the weight remains in the region $w \simeq 1$.

ing the one-point Green function. Anderson [16], Thouless [24], and Abouchacra, Anderson, and Thouless [19] have suggested looking at the imaginary part of the Green function $G_{ii}(z+i\epsilon)$: its distribution over the sites should allow one to distinguish between extended and localized states (see also Refs. [22,25,20]). We propose here an alternative way to look at the localization using $G_{ii}(z)$ but again *keeping z real* (a discussion within the language of [19] is given in Appendix C). The localization criterion becomes unambiguous: it is given in terms of the distribution of the $\{w_{i\alpha}\}$ over the sites. The derivation of the equations which allow one to determine the localization threshold is rather transparent. Our final equations coincide with those obtained in [19] after some approximations, recalled in Appendix C.

We have shown in the preceding section that the distribution of eigenvalues was given by the tail of the distribution of $G_{ii}$: $P(G_{ii}) \sim \rho(z)/G_{ii}^2$. Let us now define (for finite $N$)

$$F_{ii} \equiv -\frac{\partial G_{ii}}{\partial z} = \sum_{i=1}^{N} \frac{w_{i\alpha}}{(z-E_\alpha)^2} \ . \tag{23}$$

Similarly to the analysis of Eq. (13), one can show, using Appendixes A and B, that the tail of the distribution of $F$ is given by

$$P(F_{ii}) \underset{F_{ii}\to\infty}{\sim} \rho(z) N \overline{(w)^{1/2}} / F_{ii}^{3/2} \ , \tag{24}$$

where the average of $\sqrt{w}$ is taken over the sites $|i\rangle$, or equivalently, at fixed $z$: $(w)^{1/2} = \int_0^1 dw\, P(w/z)\sqrt{w}$. The value of $\Upsilon \equiv N\overline{(w)^{1/2}}$ is another indicator of the "localization" of a state $|\alpha\rangle$ with eigenvalue $E_\alpha = z$: if $|\alpha\rangle$ is strongly delocalized, each $w$ is of order $1/N$ and $\Upsilon$ is of order $\sqrt{N}$ which diverges for infinite $N$. If on the other hand $|\alpha\rangle$ is strongly localized over $p$ sites, $w$ is of order $1/p$ on $p$ sites and essentially zero on all the others. Then $\Upsilon \sim \sqrt{p}$ remains finite for infinite $N$. Hence the knowledge of the tail of the distribution of $F_{ii}$ allows one to know if the states are localized in the above sense, which, as we shall show, coincides with the usual definition of localization in terms of the imaginary part of the self-energy. It is, however, clear that the inverse participation ratio $Y$ can be finite while $\Upsilon$ diverges for large $N$, leading to "mixed" states: this would be the case if a finite fraction of the weight is concentrated on $p$ sites. It is also the case when the distribution $P(w)$ is a power law as in Eq. (22), but with $\frac{1}{2} < \nu < 1$ (see Appendix B). On the other hand, for $\nu < \frac{1}{2}$, and in the particular case of exponentially localized states $\nu = 0^+$, one indeed finds $\Upsilon < \infty$.

One can easily obtain a recursion equation for $F_{ii}$ by taking the derivative of Eq. (7) with respect to $z$. This leads to the coupled set of equations

$$\frac{1}{G_{00}} = z - \sum_{i=1}^{N} H_{0i}^2 G_{ii} \ , \tag{25a}$$

$$\frac{F_{00}}{G_{00}^2} = 1 + \sum_{i=1}^{N} H_{0i}^2 F_{ii} \ . \tag{25b}$$

It is easier to solve this set of equations in terms of the self-energy $S_0$ and its derivative $D_0 = -\partial S_0/\partial z$. We find

$$S_0 = \sum_{i=1}^{N} \frac{H_{0i}^2}{(z-S_i)} \ , \tag{26a}$$

$$D_0 = \sum_{i=1}^{N} \frac{H_{0i}^2 (1+D_i)}{(z-S_i)^2} \ . \tag{26b}$$

These equations are exactly those obtained by Abouchacra, Anderson, and Thouless [19], if $S_0$ and $D_0$ are identified as the real and the imaginary part of the self-energy, in the limit where this imaginary part is much smaller than the real part, i.e., in the localized regime. (See Appendix C.) Note that (26a) gives back the distribution of $P_S(S)$ found above [Eq. (10)]. The range of energies such that Eqs. (26) lead to well defined solutions thus corresponds to localized states in the sense that $\Upsilon < \infty$.

The localization criterion in terms of $D_i$ is the same as for $F_{ii}$, since $F_{ii}/G_{ii}^2 = 1 + D_i$. We immediately see that if $\Upsilon$ is infinite, $F_{ii}$ and $D_i$ are infinite with probability 1. The converse is also true.

As mentioned above, Eq. (22) implies that the inverse participation ratio is finite as soon as $\nu < 1$, while $\Upsilon < \infty$ only if $\nu < \frac{1}{2}$. This second criterion is stronger than the first and we expect, from our numerical result $\nu \sim \mu/2$, that while the first one is satisfied for $\mu < 2$ and $z > z_c$, the second can only be satisfied for $\mu < 1$. This can be seen directly by looking at the recursion relation:

$$D_0 = \sum_{i=1}^{N} H_{0i}^2 F_{ii} \ . \tag{27}$$

By construction, the variables $\{F_{ii}\}$ and $\{H_{0i}\}$ are uncorrelated (see Sec. I). Then we can use the generalized central limit theorem (Appendix A) for $D_0$. We find that $D_0$ is distributed according to a Lévy distribution $L_{\mu/2}^{C,\beta=1}(D_0)$ for $\mu < 1$ and is infinite with probability 1 for

$\mu>1$: hence *there is no localized state with* $\Upsilon<\infty$ *for* $\mu>1$.

We will closely follow the steps taken in Ref. [19] in order to solve the above recursion equations. These equations are quite complicated because $S_0$ and $D_0$ are correlated variables (they are related to the same set of $\{H_{0i}\}$). Let $Q(S_0,D_0)$ denote their joint probability distribution. We have seen above that $P_D(D_0)\equiv\int dS\,Q(S,D_0)$ should be a Lévy distribution $L_{\mu/2}^{C,\beta=1}(D_0)$. A reasonable assumption for the tail of the joint distribution $Q(S,D_0)$ is thus

$$Q(S,D_0)\underset{D_0\to\infty}{\sim}\frac{A(S)}{D_0^{1+\eta}}\ , \tag{28}$$

which can be expected to hold beyond a certain crossover, which is an increasing function of $S$. Integrating over $S$ can only slow down the decay of the distribution of $D$, and thus $\eta$ is constrained to be $\geq\mu/2$.

If we now suppose as above that the distribution of $(S^{N+1},D^{N+1})$ is the same as that of $(S^N,D^N)$, we can write (26) as an equation for $Q(S,D)$:

$$\hat{Q}(k,k')=\left[\int dS\,P(H)\hat{Q}_2\left[S,\frac{k'H^2}{(z-S)^2}\right]\times e^{-ikH^2/(z-S)-ik'H^2/(z-S)^2}\right]^N , \tag{29}$$

where $\hat{Q}$ means the double Fourier transform and $\hat{Q}_2$ the Fourier transform over the second variable. In the limit of $k'$ going to zero, one can use the asymptotic form of $Q$ [Eq. (28)], which reads, in Fourier space, $\hat{Q}_2(S,k')=\hat{Q}_2(S,0)+a|k'|^\eta a(S)$ ($a$ is a certain coefficient) and expand Eq. (29) to lowest order in $k'$. If, as found below, $\eta<1$, the resulting equation depends on the two parameters $z$ and $\eta$ and reads

$$\hat{A}(k)=\hat{L}_{\mu/2}^{C(z),\beta(z)}(k)\int_{-\infty}^{+\infty}dS\frac{A(S)}{|z-S|^{\eta+\mu/2}}|k|^{\mu/2-\eta}\Gamma\left[\eta-\frac{\mu}{2}\right]\exp\left[-i\frac{\pi}{2}\left[\eta-\frac{\mu}{2}\right]\mathrm{sgn}\{k(z-S)\}\right] . \tag{30}$$

A necessary requirement for states to be localized in the sense that $D$ remains finite when $N\to\infty$ is that there exists an $\eta>\mu/2$ which allows one to solve this "asymptotic" equation. The value $z_c'$ below which the solution disappears will be the mobility edge in that sense (or at least a lower bound of the mobility edge).

We shall find that the simplified equation (30) admits solutions only for $\mu/2<\eta<1-\mu/2<1$. Introducing the two variables

$$X=\int_{-\infty}^{z}dZ\frac{A(S)}{|z-S|^{\eta+\mu/2}}\ , \tag{31a}$$

$$Y=\int_{z}^{+\infty}dS\frac{A(S)}{|z-S|^{\eta+\mu/2}}\ , \tag{31b}$$

we obtain from Eq. (31) a system of two linear equations:

$$X=x_1X+x_2Y\ ,\quad Y=y_1X+y_2Y\ , \tag{32}$$

with

$$\begin{aligned}x_1&=\sin(\pi\mu/2)\Theta_c+\cos(\pi\mu/2)\Theta_s\ ,\\ x_2&=\sin(\pi\eta)\Theta_c+\cos(\pi\eta)\Theta_s\ ,\end{aligned} \tag{33a}$$

$$\begin{aligned}y_1&=\sin(\pi\eta)\Theta_c-\cos(\pi\eta)\Theta_s\ ,\\ y_2&=\sin(\pi\mu/2)\Theta_c-\cos(\pi\mu/2)\Theta_s\end{aligned} \tag{33b}$$

and

$$\Theta_c=\Gamma(\eta-\mu/2)\Gamma(1-\eta-\mu/2)\times\int_0^{+\infty}\frac{du}{\pi}e^{-C(z)u}\times\cos\left[+zu^{2/\mu}-C(z)u\beta(z)\tan\left[\pi\frac{\mu}{2}\right]\right] , \tag{34a}$$

$$\Theta_s=\Gamma(\eta-\mu/2)\Gamma(1-\eta-\mu/2)\times\int_0^{+\infty}\frac{du}{\pi}e^{-C(z)u}\times\sin\left[+zu^{2/\mu}-C(z)u\beta(z)\tan\left[\pi\frac{\mu}{2}\right]\right] . \tag{34b}$$

This system will have nontrivial solutions if its determinant $(x_1-1)(y_2-1)-y_1x_2$ equals zero, thereby determining $\eta$ as a function of $z$. We find that two solutions indeed exist for $|z|$ larger than a certain threshold $z_c'(\mu)$, which is plotted in Fig. 3: $z_c'$ is an increasing function of $\mu$, with $z_c'(0)=0$ and $z_c'(1)=+\infty$. As $z$ reaches $z_c'$, the two solutions merge into $\eta(z_c')=\frac{1}{2}$. Since one expects $D_0$ (i.e., the imaginary part of the self-energy) to be smaller and smaller as $z$ increases, the correct solution for $z>z_c'$ should be $\eta_+>\frac{1}{2}$. Note that $\eta_+(z\to\infty)=1-\mu/2$.

We can find a simple interpretation of the exponent $\eta_+$ in terms of the distribution of the $\{w_{\alpha i}\}$ discussed above. We have seen that if $G_{00}$ is very large it is dominated by its largest term, i.e., $w/(z-E_\alpha)$. In the localized phase ($z>z_c'$), this very large term correspond to $w$ of order 1 (for infinite $N$), and $z-E_\alpha\ll1$ but still of order 1 (and not, say, $N^{-1}$: see Appendix B). Then $F_{00}$ will also be very large and is given by $-w/(z-E_\alpha)^2=-G_{00}^2/w$. Thus we find that $1/w=-F_{00}/G_{00}^2\equiv1+D_0$. $G_{00}$ large means by definition that $S_0\simeq z$. The probability to find a large $D_0$ with $S_0\sim z$ is thus $Q(z,D_0)\sim A(z)/(D_0^{1+\eta_+})$. This must be compared to the probability that $G_{00}$ is large for a fixed small $w$, which reads $P(w)w\rho(z)$ (since $E_\alpha-z$ must be smaller than $w$). The change of variable $w=D_0^{-1}$ finally allows one to find a relation between $\nu$ [see Eq. (22)] and $\eta_+$, namely, $\nu=1-\eta_+(z)$. Hence $\nu$ is

equal to $\frac{1}{2}$ for $z=z_c'$ and decreases to $\mu/2$ when $z$ increases, in qualitative agreement with our numerical data for $\mu<1$. For $\mu>1$, however, we have not found a way to compute the exponent $\nu$.

Let us conclude this rather dense section by summarizing our results (see also Fig. 3). We have found the following.

(1) For $\mu>1$, there exists a critical value $z_c$ of the energy separating extended states with $Y=0$ from "localized" states with $Y>0$. These localized phases are , however, not of the usual type since $\Upsilon$ is still infinite (i.e., the imaginary part of the self-energy does not vanish on the real axis), corresponding to algebraically localized states with $\nu>\frac{1}{2}$. In more physical terms, this means, following Thouless and Miller and Derrida [24,20], that the current flowing from an arbitrary input point is not zero (i.e., the conductivity is finite).

(2) For $\mu<1$, all the states (except the zero-energy states) have a finite participation ratio. However, only those for which $z>z_c'$ are "strongly" localized in the usual sense that the current flowing from an input site is zero ($\Upsilon$ is finite). For $z<z_c'$, a finite fraction of the wave functions is in some sense "delocalized," i.e., is scattered over $O(N)$ sites. We have obtained an analytical expression of $z_c'(\mu)$ which diverges for $\mu=1$, and computed the exponent $\nu$ which describes the weight distribution function. Let us now see how these properties are reflected in the level spacing distribution.

### D. Distribution of level spacings

Another quantity of great theoretical (and experimental) interest is the so-called level spacing distribution, giving information on the spectrum at the smallest scale, i.e., $1/N$. Define the level spacing as the difference between successive eigenvalues $E_{\alpha+1}-E_\alpha$. Its mean value gives the local density of states $[N\rho(E_\alpha)]^{-1}$. Then the interesting quantity to consider is the distribution of the difference between successive eigenvalues normalized to 1, i.e., $s_\alpha=N\rho(E_\alpha)(E_{\alpha+1}-E_\alpha)$, for all the eigenvalues such that $Z<E_\alpha<z+dz$. Generally, this distribution $P_s(s)$ depends on $z$. In the case of exponentially localized states, an eigenvector typically spreads over a finite number of $p$ sites. The overlap between two such eigenvectors corresponding to successive eigenvalues will typically decrease exponentially as a function of the size $N$ of the system. The eigenvalues are thus uncorrelated and the level spacing distribution obeys the Poisson law $P_p(s)=\exp(-s)$. In the case of GOE matrices, all the states are extended, and hence level crossings are forbidden: the eigenvalues are strongly correlated by this level repulsion and $P_s(s)$ is well described by the "Wigner surmise" $P_w(S)=(\pi/2)s\exp-\pi s^2/4$. This distribution is expected to be to a large degree universal and appears in many situations (extended chaotic systems, transfer matrix of weakly disordered metals, etc.).

Very recently, however, there has been quite a number of alternative suggestions arising from the studies of sparse banded matrices [26,27], billiards [28], or the localization transition [29–31]. It is thus of interest to discuss this distribution in the case of Lévy matrices. The results once again depend on $\mu$.

(1) For $\mu<1$, the situation is relatively simple: as shown in Fig. 7, either $z<z_c'$ and $P_s(s)$ is of the Wigner type [this is expected since a fraction of the state is "extended," i.e., lives on $O(N)$ sites], or $z>z_c'$ and a Poisson distribution is found, reflecting "strong" localization ($\Upsilon<\infty$).

(2) For $\mu>1$, however, we find that the distribution is very close to the Wigner surmise for $z<z_c$ (Fig. 8). For $z>z_c$ the distribution is *nonuniversal* and depends continuously on $z$, which might be related to the unusual (power-law) nature of the localized states with $\Upsilon=\infty$. As expected, it is intermediate between Poisson's law and the Wigner surmise, with $P_s(s)$ rising from 0 for small $s$ much more abruptly than $\pi s/2$. Poisson's law is only recovered for $z$ going to infinity.

We have tried to fit our numerical $P_s(s)$ with the so-called "Brody" distribution $\sim s^\beta\exp(-s^{\beta+1})$ [32] without sources. This is surprising in view of the reported statistical robustness of this distribution in other contexts [26]. Two other possibilities could be explored. One could, following Pichard and Shapiro [30], propose that the "Coulomb" logarithmic interaction in the Dyson gas language is progressively screened out by localization effects, leading to a nonuniversal $P_s(s)$ interpolating between the Wigner surmise in the absence of screening and the Poisson distribution for complete screening (see also [33]). However, contrarily to the unitary case, the analytic form of $P_s(s)$ for intermediate cases is not known and a Monte Carlo calculation would be needed for a detailed comparison.

Another interesting route is the "Brownian" model of Dyson, Pechukas, and Yukawa [34,33], who propose to study the eigenvalue statistics by following their "position" as a function of "time," corresponding to the addi-

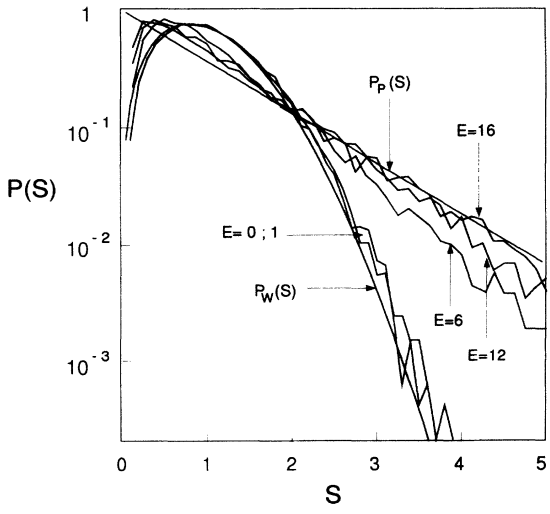


FIG. 7. Level spacing distribution for $\mu=0.5<1$, $N=800$, which is well fitted by the Wigner distribution $P_w(s)$ for $E<z_c'$ and by the Poisson distribution $P_p(s)$ for $E>z_c'$. Note that $E=6$ is close to the transition, and could correspond to the distribution studied in [29].

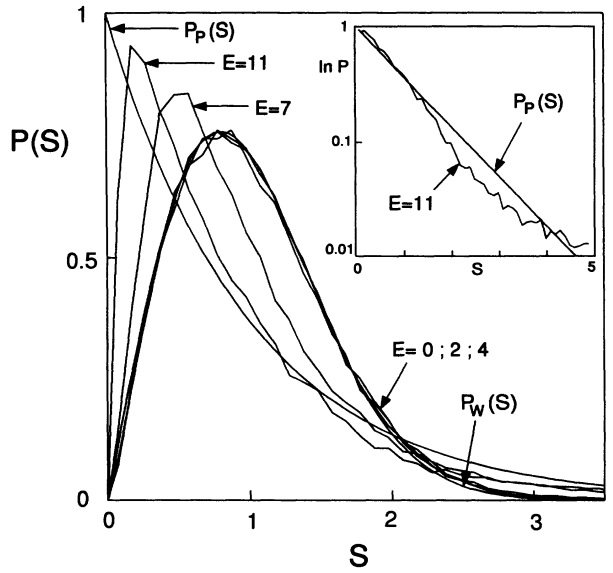


FIG. 8. Level spacing distribution for $\mu=1.5>1$, $N=300$. For $E<z_c$, $P_s(s)$ is again well fitted by the Wigner distribution. For $E>z_c$, however, $P_s(s)$ departs significantly from a Poisson distribution (see inset), which is only reached in the limit $E\to\infty$. This persists for larger values of $N$. Note that the Brody distribution was not found to be an adequate representation of $P_s(s)$ in that region.

tion of another "slice" of random matrix. This leads to a Langevin-like equation with a potential term describing the level repulsion and a noise term defining the "temperature." One can argue that in the Lévy case, this noise term will acquire an infinite variance. Interestingly, very little is known about the problem of "Lévy flights" in the presence of an external force, and in particular about the generalization of the Boltzmann equilibrium distribution (see [35]). We have shown [36] that in the case of a single particle in a quadratic well, the equilibrium distribution is a Lévy law of width related to the curvature of the well. This can be interpreted as the superposition of Boltzmann weights with a broad distribution of temperatures, decaying as $T^{-1-\mu/2}$, such that $\langle T\rangle=\infty$. Generalizing this result to the logarithmic repulsion term which appears in the present level spacing problem, one may argue that the small $S$ behavior of $P_s(s)$ should be of the form $P_s(s)\simeq(\log s)^{-\mu/2}$. This gives rise to a very abrupt rise of $P_s(s)$, qualitatively consistent with our numerical observations. Clearly, however, more work is needed to clarify this point.

## IV. CONCLUSION

Let us summarize the main points of our work. We have introduced a family of random matrices, motivated both by rather mathematical considerations and some specific physical situations. Our ensemble of Lévy matrices generalizes the GOE in the same sense as the Lévy distributions generalize the Gaussian. We have obtained analytically the density of states for such matrices, and confirmed our calculations by numerical simulations. Our analytical approach is somewhat original since we chose to work only with real (but not self-averaging) quantities. The tail of the density of states strongly suggests that the high-energy states are localized, and that a localization transition might occur. We have demonstrated the existence of such a transition numerically by studying the inverse participation ratio and analytically through the study of the one-point Green function. Interestingly, the location of the transition is different for these two quantities, suggesting the existence of a mixed phase where the states are *both localized and extended, depending on one's standpoint.* The situation is summarized by the phase diagram of Fig. 3. Finally, we have discussed the level spacing distribution, which is found to be *nonuniversal* in a certain regime, and have proposed possible explanations for these numerical findings.

In view of the growing application field of Lévy distributions in physics, we hope that the understanding of wildly fluctuating random matrices will also prove useful—for example, in economical or financial theories [37].

## ACKNOWLEDGMENTS

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## APPENDIX A: LÉVY DISTRIBUTIONS

### Definitions

A centered Levy distribution $L_\mu^{C,\beta}$ is known through its Fourier transform $\hat{L}(k)$:

$$L_\mu^{C,\beta}(x)=\frac{1}{2\pi}\int dk\,\hat{L}(k)e^{ikx}\,, \tag{A1}$$

$$\ln\hat{L}(k)=-C|k|^\mu[1+i\beta\,\mathrm{sgn}(k)\tan(\pi\mu/2)]\,. \tag{A2}$$

The three parameters $\mu$, $C$, and $\beta$ are easily interpreted if we study the tails of this distribution:

$$L_\mu^{C,\beta}(x)\underset{x\to\pm\infty}{\sim}\frac{C_\pm}{|x|^{1+\mu}}\,, \tag{A3}$$

with $C_+$ and $C_-$ given by the relations

$$\beta=\frac{C_+-C_-}{C_++C_-}\,,\quad C=\gamma_\mu(C_++C_-)\,. \tag{A4}$$

$\mu$ is the exponent describing the decay far in the tail, and for $0<\mu<2$ the distribution has an infinite variance. The mean value is finite only for $\mu>1$. $\beta$ characterizes the asymmetry of the law: $-1\le\beta\le+1$. If $\beta=\pm1$, $C_\mp=0$ and the law is maximally disymmetric and if $\beta=0$, the law is symmetrical. $C$ is a "generalized variance" in the sense that the typical value of $x$ is $C^{1/\mu}$ ($\gamma_\mu$ is a $\mu$-dependent numerical coefficient).

### Attraction basin of the Lévy distributions

There exists for infinite variance distributions an equivalent of the central limit theorem for finite variance

distributions. Consider a set of independent variables $\{x_i\}_{i=1,N}$ distributed according to a power-law distribution $P$:

$$P(x) \underset{x\to\pm\infty}{\sim} \frac{C_\pm}{N|x|^{1+\mu}} . \tag{A5}$$

If $S=\sum_{i=1}^{N} x_i$ then, when $N$ goes to infinity, $P(S)$ will have a stable form. If $\mu<1$, $S$ is distributed according to a Lévy distribution $P(S)=L_\mu^{C,\beta}(S)$ with $C$ and $\beta$ given by the relations (A4). If $\mu>1$ then $P(x)$ has a finite mean value $\bar{x}$, and $P(S)=L_\mu^{C,\beta}(S-N\bar{x})$, with $C$ and $\beta$ still given by the relations (A4).

A generalization of this theorem is the case where $S=\sum_{i=1}^{N} c_i x_i$ with the set $\{x_i\}_{i=1,N}$ distributed as previously, and $\{c_i\}_{i=1,N}$ is a set of variables sufficiently regular. Then in the limit where $N$ goes to infinity, and for $\mu<1$, $S$ is distributed according to a Lévy distribution $L_\mu^{C,\beta}(S)$ with

$$\beta=\frac{C_+-C_-}{C_++C_-}\left[\frac{1}{N}\sum_{i=1}^{N}\mathrm{sgn}(c_i)\right] , \tag{A6}$$

$$C=\gamma_\mu(C_++C_-)\left[\frac{1}{N}\sum_{i=1}^{N}|c_i|^\mu\right] . \tag{A7}$$

### Calculation of the inverse participation ratio for a Lévy distribution

Let us consider a set of $\{x_i, x_i\geq 0\}_{i=1,N}$ distributed according to a power law $P(x)\sim_{x\to\infty} 1/Nx^{1+\nu}$, with $\nu<2$. The inverse participation ratio associated to this set is given by (see Sec. III A)

$$Y=\frac{\sum_{i=1}^{N} x_i^2}{\left[\sum_{i=1}^{n} x_i\right]^2} . \tag{A8}$$

Applying the above theorems, one sees that the numerator is of order 1 and distributed according to a Lévy distribution $L_{\nu/2}^{C,\beta}$ with $\beta=1$ and $C=\gamma_{\nu/2}$. But the typical value of the denominator depends on the value of $\nu$. If $\nu>1$ then the distribution of $x$ has a finite mean value $\sim N^{-1}$ and the denominator equals, in the limit $N$ going to infinity, $N^2\bar{x}^2$, which is of order $N$. Then the inverse participation ratio is zero for $N$ infinite. For $\nu<1$, however, the distribution does not have any mean value and the denominator is the square of a quantity distributed as a Lévy law $L_\nu^{C=\gamma_\nu,\beta=1}$. The inverse participation ratio is then of order one, but is not self-averaging. One can explicitly make the calculations for the average quantity $\bar{Y}$ on the different sets $\{x_i, x_i\geq 0\}_{i=1,N}$ for $N$ going to infinity. One finds

$$\bar{Y}=\frac{\Gamma(2-\nu)}{\Gamma(2)\Gamma(1-\nu)}=1-\nu .$$

Note that very similar calculations appear in the spin-glass literature, for reasons discussed in [38,39]. In particular, the full distribution of $Y$ can be discussed [39].

## APPENDIX B: DISTRIBUTION OF THE LOCAL WEIGHTS

### Tail of the distribution of $G_{ii}$

The derivation of the density of states given in Sec. II implicitly assumed that $w\sim 1/N$. For localized states, however, $P(w)\sim 1/(Nw^{1+\nu})$, with $\nu<1$ and hence $w$ is typically of order $1/N^{1/\nu}$. The largest term in $G_{ii}$ [see Eq. (2)] for large $G_{ii}$ is no longer given by the smallest $z-E_\alpha$ because this term will be of order $1/N$ and so $w/(z-E_\alpha)\sim N^{1-1/\nu}$ which goes to zero because $\nu<1$ for localized states. We will show here that very large $G_{ii}$ is still dominated by one of the term $w/(z-E_\alpha)$ but with both $w$ and $z-E_\alpha$ of order 1 compared to all scales in $N$.

The total number of $\{w\}$ of order $1/N^\chi$ with $0<\chi<1/\nu$ is given by $NP(N^{-\chi})N^{-\chi}\sim N^{\chi\nu}$. On the other hand, the smallest $z-E_\alpha$ among $N^{\chi\nu}$ such terms is of order $1/N^{\chi\nu}$. Then the largest term $w/(z-E_\alpha)$ with $w\sim 1/N^\chi$ is of order $N^{\chi(\nu-1)}$ and goes to zero for all $\chi\neq 0$. Then we see that the largest term in the sum defining $G_{ii}$ is given by $w$ and $z-E_\alpha$ (small) but of order 1 when $N\to\infty$. The sum rule $\int_0^1 P(w|z)w\,dw$ equals $1/N$ is conserved even if all $w\simeq N^{-\chi}$ are discarded because, for $\nu<1$, the integral *converges* around $w=0$. Hence our central equation (14) is also true for localized states.

We can use similar arguments to show that the tail of the distribution of $F_{ii}$ is given by (24) for all the values of $\nu$.

### Inverse participation ratio

We give here more information relative to the inverse participation ratio when the $\{w\}$ are distributed according to a power law $P(w)\propto 1/w^{1+\nu}$ when $w$ in the region $N^{-1/\nu}\ll w\ll 1$. The condition of normalization of the state $\int_0^1 dw\,P(w)w=1/N$ gives

$$P(w)\sim\frac{1}{Nw^{1+\nu}} \quad \text{for } \nu<1 , \tag{B1a}$$

$$P(w)\sim\frac{1}{N^\nu w^{1+\nu}} \quad \text{for } \nu>1 . \tag{B1b}$$

In the case $\nu>1$ the integral is divergent and one must add a cutoff in this scaling form of the distribution at $1/N$ which becomes the typical value of the $\{w\}$. Translating this in real space, we find that it corresponds to volume dependent wave functions, i.e., $w(R)\simeq 1/N^{\nu-1}R^\gamma$ with $\gamma=d/\nu$, characteristic of *extended* states. If we now calculate the inverse participation ratio $Y=N\int_0^1 dw\,P(w)w^2$, using (B1), we find

$$Y\sim O(1) \quad \text{for } \nu<1 , \tag{B2a}$$

$$Y\sim O(N^{1-\nu}) \quad \text{for } 1<\nu<2 , \tag{B2b}$$

$$Y\sim O\left[\frac{1}{N}\right] \quad \text{for } \nu>2 . \tag{B2c}$$

The inverse participation ratio is then finite only for $\nu<1$. Similarly, $\Upsilon=N\int_0^1 dw\,P(w)\sqrt{w}$ is finite only when $\nu<\frac{1}{2}$.

## APPENDIX C: DISTRIBUTION OF THE IMAGINARY PART OF THE SELF-ENERGY

Anderson, Thouless, and Abouchacra *et al.* have suggested looking at the *distribution* of the imaginary part of the one-point Green function (the local density of states) in order to get some information about the localization transition. There is in fact a huge difference for this distribution for extended and localized states, even if the mean value over the sites (i.e., the density of states) is of course regular at the transition.

Let us write the imaginary part of $G_{ii}$ as

$$\mathrm{Im}(G_{ii})=\epsilon\sum_{\alpha=1}^{N}\frac{w_{\alpha i}}{(z-E_\alpha)^2+\epsilon^2}\ . \tag{C1}$$

The interesting limit is to take $N\to\infty$ first, and then the limit $\epsilon\to 0$ is taken, which means for finite $N$ and $\epsilon$: $1/N\ll\epsilon\ll 1$. Let us discuss Eq. (C1) under the assumption that the quantities $w_{\alpha i}$ and $z-E_\alpha$ are uncorrelated. The distribution of $\mathrm{Im}(G_{ii})$ thus depends on the distribution of $w_{\alpha i}$ for *a fixed site* $|i\rangle$. One should carefully distinguish this distribution from the one discussed in the text, Eq. (22), which pertains to the distribution of $w_{\alpha i}$ for a fixed $|\alpha\rangle$.

The large terms in the sum (C1) are those with $z-E_\alpha\sim O(\epsilon)$. As the other terms will only contribute to order $\epsilon$, we will neglect them as long as we are interested in large values of $\mathrm{Im}G_{ii}$. In the case of extended eigenstates ($z<z_c$) where the projection $w_{\alpha i}$ is of order $1/N$, one finds that

$$\mathrm{Im}(G_{ii})\simeq\int dE\,\rho(E)\frac{\epsilon}{(z-E)^2+\epsilon^2}=\pi\rho(z)\ ,\quad \epsilon\to 0\ . \tag{C2}$$

On the other hand, for $z>z_c$, our numerical results suggest that $w_{\alpha i}$ for a fixed $|i\rangle$ is distributed as a power law $P(w)\sim c'(z)/Nw^{1+\delta(z)}$, with $\delta(z)\simeq\frac{1}{2}$ for all $z>z_c$ (it is difficult to assert the dependence of $\delta$ on $z$). The imaginary part of $G_{ii}$ is thus the sum of the product of two random quantities, $w_{\alpha i}$ and $\epsilon/[(z-E_\alpha)^2+\epsilon^2]$ both distributed according to a power law *with an upper cutoff* ($w_{\alpha i}\le 1$ and $\epsilon/[(z-E_\alpha)^2+\epsilon^2]\le 1/\epsilon$). The distribution of $\mathrm{Im}G_{ii}$ depends on the relative value of the exponents of the tails, namely $\delta(z)$ and $\frac{1}{2}$. We find (see Appendix A)

$$P(g=\mathrm{Im}G)\sim\begin{cases}\dfrac{\rho(z)\sqrt{\epsilon}}{g^{3/2}} & \text{for } \delta<\frac{1}{2}\\[2ex] \dfrac{\rho(z)\epsilon^{1-\delta}}{g^{1+\delta}} & \text{for } \delta>\frac{1}{2}\ .\end{cases} \tag{C3}$$

This last equation means that $\mathrm{Im}G$ is of order $1/\epsilon$ or greater with probability $\epsilon$ and of order $\epsilon(\epsilon^{1/\delta-1})$ with probability $1-\epsilon$ for $\delta<\frac{1}{2}$ (respectively, $\delta>\frac{1}{2}$). The mean value over the sites of this distribution is thus finite: it must be equal to $\pi\rho(z)$. In the limit where $\epsilon$ goes to 0, $\mathrm{Im}G$ is always zero except on a finite set of points where it is infinite. It is the difference between Eqs. (B2) and (B3) which, in principle, alllows one to distinguish between extended and localized states, and which can be exploited to describe analytically the transition.

Let us rewrite Eq. (7) for the *imaginary part of the self-energy*

$$\mathrm{Im}\Sigma_0\equiv\Delta_0=\sum_{i=1}^{N}H_{0i}^2\,\mathrm{Im}G_{ii}\ . \tag{C4}$$

The variables $\{\mathrm{Im}G_{ii}\}$ and $\{H_{0i}\}$ are uncorrelated (see Sec. I). If $\epsilon>0$, $\mathrm{Im}G_{ii}$ has an upper cutoff. Then we can use the generalized central theorems for $\Delta_0$. We find that $P(\Delta_0)=L_{\mu/2}^{C,\beta=1}(\Delta_0)$, with $C=O(1)$ for extended states and $C=\epsilon^{\mu/2}$ (or $C=\epsilon^{(\mu/2)(1/\delta-1)}$) for localized states and $\delta(z)<\frac{1}{2}$ [respectively, $\delta(z)>\frac{1}{2}$]. Whether or not $\Delta_0$ tends to zero for small $\epsilon$ should allow us to find the nature of the states around a given energy $z$.

The self-consistent equation (7) is now written in terms of the real and imaginary parts of self-energy $\Sigma_i=S_i+i\Delta_i$ as

$$\Delta_0=\sum_{i=1}^{N}\frac{N_{0i}^2(\epsilon+\Delta_i)}{(z-S_i)^2+(\epsilon+\Delta_i)^2}\ , \tag{C5a}$$

$$S_0=\sum_{i=1}^{N}\frac{H_{0i}^2(z-S_i)}{(z-S_i)^2+(\epsilon+\Delta_i)^2}\ . \tag{C5b}$$

It is difficult to find the stable distribution of $\Sigma_0$ from these equations in the general case. However, in the case $z>z_c$ where the states are localized, one expects from the above discussion, that $\Delta$ should be of order $\epsilon$ (or $\epsilon^{1/(\delta-1)}$). Then the terms $(\epsilon+\Delta_i)^2$ can be neglected compared to $(z-S_i)^2$. It is then natural to consider, for $\delta(z)<\frac{1}{2}$, the rescaled quantity $\tilde{\Delta}=\Delta/\epsilon$, for which the rescaled equations read

$$\tilde{\Delta}_0=\sum_{i=1}^{n}\frac{H_{0i}^2(1+\tilde{\Delta}_i)}{(z-S_i)^2}\ , \tag{C6a}$$

$$S_0=\sum_{i=1}^{N}\frac{H_{0i}^2}{z-S_i}\ . \tag{C6b}$$

These equations are exactly the equations obtained by our method (see text) considering $z$ real and looking at the tail of the distribution of the derivative of $G_{00}$. This equivalence is in fact a consequence of the analyticity of $G_{00}(z)$ [24].

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